Lecture 25

Monday, 28 November 2022 1:30 PM

Fair Division with Indivisible Items

- Set M of m items
- n agents, agent i has value fn. $v_i : 2^M \to \mathbb{R}_+$

  (I) additive if $v_i(S) = \sum_{j \in S} v_i(j) \quad \forall S \subseteq M, i$

  (II) monotone if $v_i(X \cup a) \geq v_i(X) \quad \forall i, a, X$
- allocation $A = (A_1, \dots, A_n)$ s.t. $\forall i, A_i \subseteq M$

  $A_i \cap A_j = \phi \; \forall i, j$

  $\bigcup_i A_i = M$

  (if last property is violated, then this is a partial allocation)

An allocation $A = (A_1, \dots, A_n)$ is Envy-Free if $\forall i, j$, $v_i(A_i) \geq v_i(A_j)$

Easy to see there may not $\exists$ an E-F allocation, so we consider relaxations?

Lemma: It is NP-hard to determine if a given instance admits an EF allocation, even with 2 agents, additive, identical valuations (i.e., $v_i(S) = v_j(S) \; \forall S \subseteq M$)

(prove yourself)

In the following, we assume additive valuations.

Defn: An allocation A is envy free upto one item (EF1) if
$\forall i,j,\ v_i(A_i) \geq v_i(A_j \setminus x)$ for some $x \in A_j$

Does an EF1 allocation always exist? Yes!
Consider the Round Robin (RR) algo:

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- Order the agents arbitrarily.
- $A_i \leftarrow \emptyset$ $\forall i$,   $M' \leftarrow M$
- while (1)
        |  - for $i = 1 \dots n$
        |  - if $M' = \emptyset$, break.
  round |  - let $j \leftarrow \arg\max_{k \in M'} v_i(k)$
        |
        |  - $A_i \leftarrow A_i \cup \{j\}$,  $M' \leftarrow M' \setminus \{j\}$
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Lemma: The allocation A obtained by RR is EF1.

Proof: Consider agents i, j s.t. $i < j$ (so agent i picks before agent j in each round).
Let a_i^t, a_j^t be the items picked by i, j in round t.

Then $\forall t$, $v_i(a_i^t) \geq v_i(a_j^t)$, since i picks first.

$\Rightarrow v_i(A_i) \geq v_i(A_j)$

Further, $v_j(a_j^{t-1}) \geq v_j(a_i^t)$ since a_i^t was available when j picked a_j^{t-1}

$\Rightarrow v_j(A_j) \geq v_j(A_i \setminus a_i^1)$

hence, A is EF1. $\blacksquare$

Claim: The allocation obtained by RR may not be Pareto-Optimal

An easy example is:

	Items	
Agents	a	b
1	5	5
2	10	0

If agent 1 goes first & picks item a, agent 2 must pick item b, with 0 value. Clearly this is not PO.

One can also construct examples where all values are distinct.

Q. Construct an example with distinct values (i.e., $v_i(a) \neq v_i(b)$ for all agents i & items a, b) where RR is not PO.

So, can one obtain an allocation that is both EF1 & PO?

i.e., which is both "fair" & "efficient"?

Social Welfare

We have already seen one measure of social welfare: the utilitarian welfare $\frac{1}{n}\sum_{i=1}^{n} v_i(A_i)$

Another measure is the egalitarian welfare: $\min_i v_i(A_i)$

In fact, these are just two specific cases of the generalized p-mean welfare function, parametrized by $p \in \mathbb{R}$:

$$\left(\frac{1}{n}\sum_{i=1}^{n} v_i(A_i)^p\right)^{1/p}$$

For $p=1$, we get the utilitarian welfare

For $p \to -\infty$, we get the egalitarian welfare

(For $p > 1$, we start to get the "rich get richer" phenomenon, which is contradictory to fairness).

A very interesting case is when $p \to 0$:

Claim: $$\lim_{p \to 0}\left(\frac{1}{n}\sum_{i=1}^{n}(v_i(A_i))^p\right)^{1/p} = \left(\prod_{i=1}^{n} v_i(A_i)\right)^{1/n}$$

(prove yourself)

In which case, we get the geometric mean, known as the Nash Social Welfare. For an allocation A, we define

$$NSW(A) = \left(\prod_{i=1}^{n} v_i(A_i) \right)^{1/n}$$

Among other nice properties, there are two satisfied by the NSW:

(i) Scale - invariance.

Consider two instances of fair division, where in the second instance, for each agent i, her valuations are all scaled by a factor $\lambda_i > 0$:

$$\forall S \subseteq M, \quad v_i'(S) = \lambda_i v_i(S)$$

For NSW, this scaling does not change which allocations are better.

I.e., if w/ valuations v, $NSW(A) \geq NSW(A')$,
then w/ valuations v', $NSW(A) \geq NSW(A')$

(this is not true of the utilitarian & egalitarian welfare)

In particular, for NSW it does not matter which "currency" an agent expresses her valuation in.

(ii) Pigou - Dalton Transfer Principle.

The NSW prefers allocations that are "fairer". In particular, consider two allocations $A = (A_1, A_2, A_3, \ldots, A_n)$
& $A' = (A_1', A_2', A_3, \ldots, A_n)$

(i.e., only allocations to agents 1 & 2 change)

Suppose $v_1(A_1) + v_2(A_2) = v_1(A_1') + v_2(A_2')$

but $v_1(A_1) < v_1(A_1')$, $v_2(A_2') < v_2(A_2)$

Then $NSW(A') > NSW(A)$

(again, note that this is not true of UW or EW).

Thus, maximizing the NSW of an allocation appears to be a good measure.

Lemma: Given an instance with additive monotone valuations, If allocation A maximizes the NSW, then A is both PO & EF1.

Proof: It is clear that A is PO, since it maximizes the product of valuations.

For a contradiction, assume A is not EF1. In particular, say

$$v_i(A_i) < v_j(A_j \setminus x) \quad \forall x \in A_j$$

Let $z \in \arg\max_{x \in A_j} \dfrac{v_i(x)}{v_j(x)}$.

We will show that transferring z from A_j to A_i would increase the NSW, for a contradiction.

Let $\lambda = v_i(z)/v_j(z)$.

Then $\forall x \in A_j, \quad v_i(x) \leq \lambda\, v_j(x)$

$\Rightarrow \quad v_i(A_j) \leq \lambda\, v_j(A_j)$ —— ①

also, $v_i(A_i) < v_i(A_j) - v_i(z)$ —— ⑪

Now consider $v_i(A_i \cup 2)\, v_j(A_j \setminus 2) - v_i(A_i)\, v_j(A_j)$

$$= \big(v_i(A_i) + v_i(2)\big)\big(v_j(A_j) - v_j(2)\big) - v_i(A_i)\, v_j(A_j)$$

$$= v_j(A_j)\, v_i(2) - v_i(A_i)\, v_j(2) - v_i(2)\, v_j(2)$$

$$= \left[\frac{v_j(A_j)}{v_j(2)} - \frac{v_i(A_i)}{v_i(2)} - 1\right] v_i(2)\, v_j(2)$$

(Note that since $v_i(A_j) > v_i(A_i) \geq 0$, $\exists x \in A_j : v_i(x) > 0$. If $v_j(x) = 0$, then clearly transferring x to A_i would increase the NSW. Hence we assume $v_i(2)\, v_j(2) > 0$.)

$$> \left[\frac{v_j(A_j)}{v_j(2)} - \frac{v_i(A_j) - v_i(2)}{v_i(2)} - 1\right] v_i(2)\, v_j(2)$$

$$= \left[\frac{v_j(A_j)}{v_j(2)} - \frac{v_i(A_j)}{v_i(2)}\right] v_i(2)\, v_j(2)$$

$$\geq \left[\frac{v_j(A_j)}{v_j(2)} - \frac{\lambda\, v_j(A_j)}{v_i(2)}\right] v_i(2)\, v_j(2)$$

$$= \left[\frac{v_i(2)}{v_j(2)} - \lambda\right] v_j(A_j)\, v_j(2)$$

$$= 0$$

Since $v_i(A_i \cup 2)\, v_j(A_j \setminus 2) > v_i(A_i)\, v_j(A_j)$, transferring 2 from A_j to A_i would increase the NSW, violating the assumption that A maximizes the NSW. ∎

Thus, at least for monotone additive valuations, we know that there are allocations that are both PO & EF1.

This problem however remains open for classes beyond additive valuations (eg., for submodular valuations).

Note that, however, maximizing the NSW is NP-hard, even for 2 identical agents with additive valuations (prove yourself; reduce from subset sum).

Q: For more general valuations (eg., submodular, subadditive), does there always exist an EF1 allocation.

In fact yes; for any monotone valuations, there exists an algorithm that returns an EF1 algorithm in polynomial time.

i.e., $\forall i, S \subseteq M, x \in M, v_i(S \cup x) \geqslant v_i(S)$

The algorithm is called the Envy-Cycle Elimination Algo (ECEA). It relies on the concept of an envy-graph.

Defn: Given a (possibly partial) allocation A, the envy graph is the directed graph with vertices corresponding to agents, and directed edge (i,j) if i envies j, i.e., $v_i(A_i) < v_i(A_j)$.

ECEA:

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$A_i \leftarrow \phi$ initially $\forall i$, $M' \leftarrow M$
while $M' \neq \phi$,
    choose any item $x \in M$
    If there is a source $i$ in the envy-graph w/
            the current allocation $A$,
            then assign $x$ to $i$
            $( A_i \leftarrow A_i \cup x$, $M' \leftarrow M' \setminus x )$
    else
            while there is no directed cycle
                    there must be a directed cycle
                    (i.e., $i_1$ envies $i$ ..., $i_k$ envies $i_1$)
                    "resolve" this cycle, i.e.,
                     $\forall j = 1 \dots k-1$, $A_{i_j} \leftarrow A_{i_{j+1}}$, $A_{i_k} \leftarrow A_{i_1}$
            end while
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Lemma: For each item x, the inner while loop is executed at most n^2 times.

Proof: We show that every time a cycle C is resolved, the outdegree for each agent $i \in C$ goes down by at least 1, while the outdegree remains unchanged for agents $i \notin C$.

This is easy to see, since for an agent i, if i's allocation is unchanged and the other bundles are simply redistributed, then i's outdegree is unchanged. Hence for $i \notin C$, the outdegree is unchanged.

For $i \in C$, we note that the envy towards the bundle allocated to it's following vtx. in the cycle goes away after cycle resolution. Edges to bundles of higher value remain.. Hence the out degree is decreased by at least 1.

Since there are at most n^2 edges, the inner while loop is executed at most n^2 times. ▨

Lemma: For any monotone valuations, the ECEA algorithm obtains an EF1 allocation

Proof : Assume that the current $\overset{\text{partial}}{\wedge}$ allocation A is EF1.

In a step, either an item x is assigned to a source in the envy graph, or a cycle C in the envy graph is resolved. We will show that in either case, the resulting allocation (say A') is EF1.

Case I: x is assigned to agent i.

We only need to consider envy of and by agent i.

If j envies i after allocating x, since i was a source earlier, clearly $v_j(A_i' \setminus x) \leq v_j(A_j)$, and hence this envy is removed.

If i envies j after allocating x, $\exists\, y \in A_j$ s.t.

$$v_i(A_i) \geq v_j(A_j \setminus y)$$

and hence, by monotonicity,

$$v_i(A_i \cup x) \geq v_j(A_j \setminus y)$$

Hence, in this case the resulting allocation is EF1.

Case II: The cycle C is resolved.

Consider an envy edge (i,j) after resolution of C.

If $i,j \notin C$, clearly nothing has changed.

If $i \in C$, let j' be the agent that earlier held A'_j, i.e.,

$$A'_j = A_{j'}$$

Note that $\exists\, y \in A_{j'}$ s.t.

$$v_i(A_i) \geq v_i(A_{j'} \setminus y)$$

and hence $v_i(A_i') \geq v_i(A_i) \geq v_i(A_{j'} \setminus y) = v_i(A'_j \setminus y)$

If $i \notin C$, $j \in C$, the logic of the previous case still holds,

& $\exists\, y \in A'_j$ s.t. $v_i(A_i') \geq v_i(A'_j \setminus y)$

Hence, the modified allocation A' is EF1 $\blacksquare$